It is well known, for instance, that the oblateness of the earth produces, among other effects, a secular increase in both Ω and ω (see Kozai[15] as an example):

$$d\Omega/dt = -(2\pi\epsilon\bar{R}\ ^2/p^2P)\ \cos i$$

$$d\omega/dt = (\pi\epsilon\bar{R}\ ^2/p^2P)(5\ \cos\quad - 1)$$

where p is the focal parameter of the orbit, $\bar{R}_e$ the equatorial radius of the earth, P the orbital period, i the inclination of the orbit to the equatorial plane, and ϵ a dimensionless parameter that characterizes the oblateness of the earth ($\epsilon =$ 1 6331 $\times 10^{-3}$)

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Scattering and Absorption of Plane Waves by Cylindrically Symmetrical Underdense Zones

F A Albini*

Institute for Defense Analyses, Washington, D C

Scattering and absorption of plane electromagnetic waves by cylindrically symmetrical, but axially and radially inhomogeneous, nearly transparent zones are computed in the Born approximation Scattered fields and the cross sections are presented in analytical form in terms of the elementary angles and an integral over the scattering zone The results are intended for application to meteorite and missile trail scattering; factors for the scattered amplitude are presented analytically and graphically for several electron-concentration distributions The role of turbulence is discussed, and the locally homogeneous but anisotropic turbulence case is solved for several arbitrary correlation functions It is shown that under rather loose conditions the turbulent contribution to the scattering cross section can exceed that due to the mean distribution, and that even weak anisotropy can introduce significant aspect angle sensitivity into the cross-sectional values

Introduction

THE scattering of electromagnetic waves by inhomogeneous regions has received considerable attention in recent years, with particular emphasis on cylindrical[1–4] and spherical[5–7] zones Motivation for such work stems from the investigation of meteorite trails by radar,[8] the use of scatter from ionized zones as an aid to radio communication, and the scattering of waves by the trails of hypersonic vehicles [9]

The analysis of such scattering processes and explanation of observed data are by no means closed subjects [10–12] There is some evidence that magnetic anisotropy even may be important in the scattering from meteor trails at lower frequencies[13] (a complication which readily can be included in the method of analysis here) [14] A general procedure for handling inhomogeneous scattering problems is clearly needed, and this extension of previous work[4] constitutes another step in the exploitation of the Born approximation

By reformulating the problem of the cylindrical scatterer to include axial gradients in "dielectric constant," the prob

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* Senior Technical Staff Member AIAA

lem of the finite, homogeneous cylinder is thus included as a special case It is presumed, on the basis of the previous demonstration,[4] that the results are the asymptotic representation for the exact solution in the nearly transparent limit

Formulation

Figure 1 illustrates the geometry of the problem A mixture of coordinate systems will be employed A Cartesian coordinate system is constructed with the z axis along the cylindrical axis of symmetry and the x axis in the plane of incidence This coordinate system is used to express the incident electromagnetic field components and as a reference A vector to the point of observation at infinity from the origin of the Cartesian system is described by the spherical polar angles θ_0 and φ_0, the former being the angle between the z axis and this vector, the latter being the angle between the x axis and the projection of the vector in the plane $z = 0$ Standard cylindrical polar coordinates (ρ, φ, z) are used to carry out the integrations over the cylindrical zone

The incident wave makes an angle θ with the axis of symmetry (z) Polarization of the incident wave is described by the angle (α) of the electric vector with respect to the normal to the plane of incidence For the moment, however, it is only necessary to assign a direction vector indicating the polarization, say $\mathbf{e}_p$:

$$\mathbf{e}_p = \sin\alpha \cos\theta \, \mathbf{e}_x + \cos\alpha \, \mathbf{e}_y - \sin\alpha \sin\theta \, \mathbf{e} \tag{1}$$

The incident electromagnetic wave is described by an electric field of unit intensity:

$$\mathbf{E}_i = \mathbf{e}_p \exp[ik_0(x \sin\theta + z \cos)] = \mathbf{e}_p \exp(ik_0\mathbf{e}_i \, \mathbf{r}) \tag{2}$$

where k_0 is the vacuum wavenumber, $\mathbf{e}_i$ the direction of propagation, and time dependence of the form $\exp(-i\omega t)$ is understood The associated magnetic wave is then

$$\mathbf{H}_i = (k_0/\omega\mu_0)\mathbf{e}_i \times \mathbf{e}_p \exp(ik_0\mathbf{e}_i \, \mathbf{r}) \tag{3}$$

and μ_0 is the permeability of vacuum

The scattering region is characterized by a radially symmetrical dielectric "constant" distribution everywhere near unity, i e ,

$$K(\rho,z) = 1 - \eta(\rho,z) \tag{4}$$

where

$$|\eta| \ll 1 \tag{5}$$

The propagation equation satisfied by the electric vector is

$$(\nabla^2 + k_0^2)\mathbf{E} = k_0^2\eta\mathbf{E} - \mathrm{grad}\{\mathrm{grad}[\ln(1 - \eta)] \, \mathbf{E}\} \tag{6}$$

and the scattering region is confined by the surface on which η goes to zero Call this surface S

Equation (6) is solved, under the Born approximation, by Green's integral with the fields on the right-hand side approximated by the incident field This can be looked upon as an iteration or a perturbation[15] procedure, but the process is virtually never carried beyond the first step

The Green's function for scattered waves consists of the identity dyad multiplied by the scalar Helmholtz equation, Green's function,[16] and so the solution to (6) can be written as

$$\mathbf{E}(\mathbf{r}) = -\iiint (k_0^2\eta\mathbf{E} - \mathrm{grad}\{\mathrm{grad}[\ln(1 - \eta)] \, \mathbf{E}\}) \times \frac{\exp(ik_0|\mathbf{r} - \mathbf{r}'|)}{4\pi|\mathbf{r} - \mathbf{r}'|} \, d^3\mathbf{r}' \tag{7}$$

Solution for the Far Fields

Equation (7) can be simplified by passing to the limit as $\mathbf{r}$ (the vector to the point of observation) approaches infinity Thus, if $\mathbf{e}_0$ is a unit vector along $\mathbf{r}$,

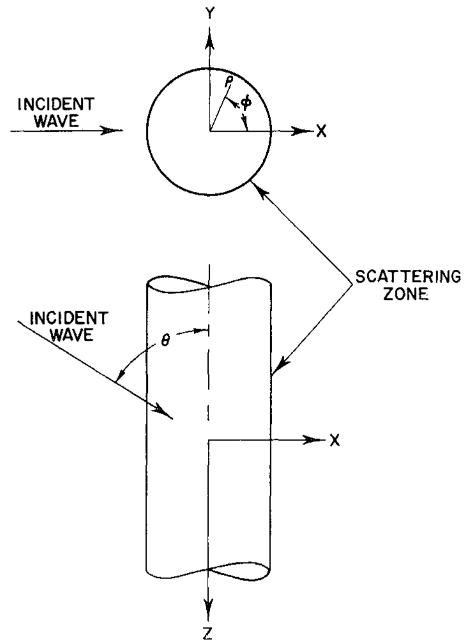

Fig 1 Geometry of scattering problem; end view above, plan view below

$$\mathbf{E}(\mathbf{r}) \to -\frac{\exp(ik_0r)}{4\pi r} \iiint \exp(-ik_0\mathbf{e}_0 \, \mathbf{r}') \times (k_0^2\eta\mathbf{E} - \mathrm{grad}\{\mathrm{grad}[\ln(1 - \eta)] \, \mathbf{E}\}) d^3\mathbf{r}' \tag{8}$$

Using Gauss' theorem in the form

$$\iiint (\mathrm{grad}f) dV = \iint_s f \, ds \tag{9}$$

permits reduction of the second term[15]:

$$\iiint \exp ik_0\mathbf{e}_0 \, \mathbf{r}')\mathrm{grad}f d^3\mathbf{r}' = \iint_s f \exp(ik_0\mathbf{e}_0 \, \mathbf{r}')ds - ik_0\mathbf{e}_0 \iiint f \exp(ik_0\mathbf{e}_0 \, \mathbf{r}')d^3\mathbf{r}' \tag{10}$$

The surface integral vanishes, as η is zero on the surface, so that

$$\mathbf{E}(\mathbf{r}) \to -\frac{\exp(ik_0r)}{4\pi r} \iiint \exp(-ik_0\mathbf{e}_0 \, r') \times \{k_0^2\eta\mathbf{E} - ik_0\mathbf{e}_0 \, \mathrm{grad}[\ln(1 - \eta)] \, \mathbf{E}\} d^3\mathbf{r}' \tag{11}$$

Invoking the Born approximation now permits a further reduction of the second term, since

$$\iiint \{\mathrm{grad}[\ln(1 - \eta)] \, \mathbf{e}_p\} \exp[ik_0(\mathbf{e}_i - \mathbf{e}_0) \, \mathbf{r}']d^3\mathbf{r}' = \mathbf{e}_p \, \{\iint \ln(1 - \eta) \exp[ik_0(\mathbf{e}_i - \mathbf{e}_0) \, \mathbf{r}']ds - ik_0(\mathbf{e}_i - \mathbf{e}_0) \iiint \ln(1 - \eta) \exp[ik_0(\mathbf{e}_i - \mathbf{e}_0) \, \mathbf{r}']d^3\mathbf{r}'\} \tag{12}$$

Again the surface integral vanishes, and since $\mathbf{e}_p \perp \mathbf{e}_i$, the right-hand side of (12) is simply

$$ik_0\mathbf{e}_p \, \mathbf{e}_0 \iiint \ln(1 - \eta)\exp[ik_0(\mathbf{e}_i - \mathbf{e}_0) \, \mathbf{r}']d^3\mathbf{r}' \tag{13}$$

so Eq (11) becomes

$$\mathbf{E}(\mathbf{r}) \to -\frac{\exp(ik_0r)}{4\pi r} k_0^2 \iiint [\eta\mathbf{e}_p + \ln(1 - \eta)\mathbf{e}_0(\mathbf{e}_p \, \mathbf{e}_0)] \exp[ik_0(\mathbf{e}_i - \mathbf{e}_0) \, \mathbf{r}']d^3\mathbf{r}' \tag{14}$$

To make this equation strictly consistent it is necessary to expand the logarithm term, using (5), so that finally,

$$E(\mathbf{r}) \to -k_0^2[\mathbf{e}_p - \mathbf{e}_0(\mathbf{e}_p \, \mathbf{e}_0)] \frac{\exp(ik_0r)}{4\pi r} \iiint \eta \exp[ik_0(\mathbf{e}_i - \mathbf{e}_0) \, \mathbf{r}']d^3\mathbf{r}' \tag{15}$$

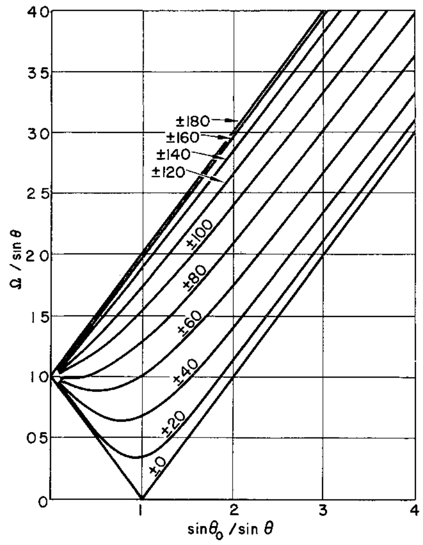

Fig 2 **Geometrical factor in the argument of J_0, Eq (20); parameter is angle ϕ_0 in degrees**

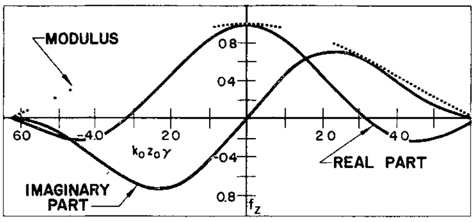

Fig 3 **Scattering factor f_z for homogeneous axial distribution**

From the form of (15) it is clear that had the second term been neglected in (7) the fields would not be transverse, since the second term contributes precisely the negative of the radial component of $\mathbf{e}_p$ For this reason it is not necessary to retain the second term in computing scattered power[15] since the radial component does not contribute to this quantity, but field intensity and polarization would be incorrect without the extra term

The magnetic field is derived, in the far zone, from the plane wave formula

$$\mathbf{H} = \frac{k_0}{\omega\mu_0}\,\mathbf{e}_0 \times \mathbf{E} = -\frac{k_0}{\omega\mu_0}\,k_0{}^2\mathbf{e}_0 \times \mathbf{e}_p\,\frac{\exp(ik_0r)}{4\pi r}\,I(\eta) \qquad (16)$$

Integration over the polar angle φ can be carried out† with the following result:

$$\mathbf{E}_s \rightarrow -\frac{k_0{}^2 \exp(ik_0r)}{4\pi r}\,[\mathbf{e}_p - \mathbf{e}_0(\mathbf{e}_p \cdot \mathbf{e}_0)]\,I(\eta) = -\frac{1}{2\pi}\,\mathbf{W}(r)I(\eta) \qquad (17)$$

$$I(\eta) = 2\pi \int\int \eta(\rho,z)\exp(ik_0\gamma z)J_0(k_0\Omega\rho)\rho d\rho dz \qquad (18)$$

where J_0 is a zero-order Bessel function and

$$\gamma = \cos\theta - \cos\theta_0 \qquad (19)$$

$$\Omega = (\sin^2\theta + \sin^2\theta_0 - 2\sin\theta\sin\theta_0\cos\varphi_0)^{1/2} \qquad (20)$$

This form is useful for studying the influence of various radial and axial distributions on the scattered field patterns Note that $\eta(\rho, z)$ must include a parametric dependence upon (generally) two lengths, from dimensional considerations Writing this specifically as

$$\eta(\rho,z) = \eta(\rho/r_0, z/z_0) \qquad (21)$$

† See Appendix

where r_0 and z_0 are, respectively, characteristic radial and axial distances, the expressions for the fields can be put into the form

$$\mathbf{E} \rightarrow -\mathbf{W}(r) \int\int \eta(U,V)r_0{}^2z_0 \exp(ik_0\gamma z_0V)J_0(k_0\Omega r_0U)\,UdUdV \qquad (22)$$

A further decomposition is instructive when the form of η is separable into a function of $\rho/r_0 = U$, multiplied by a function of $z/z_0 = V$

$$\eta(U,V) = A(V)\,B(U)\eta_0 \qquad (23)$$

Then, clearly,

$$\int\int \eta \exp(ik_0\gamma z_0V)J_0(k_0\Omega r_0U)\,UdUdV = \eta_0 f(k_0\Omega r_0) f(k_0\gamma z_0) = I(\eta)/2\pi r_0{}^2 z_0 \qquad (24)$$

Two considerations make this class of problems particularly useful First, in the trail of a meteorite at a very high altitude, the damping mechanism of electron-atom collisions is relatively insignificant, making η then proportional to electron concentration For artificial objects at lower altitudes, there is evidence that the collision damping frequency does not vary strongly with position in the wake,[17] again making η proportional to electron concentration In both instances radial variations in electron concentration are con-

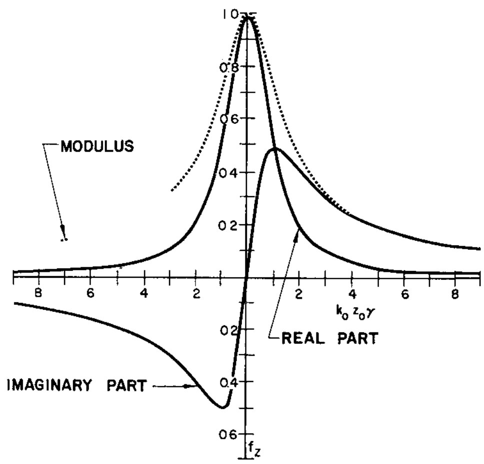

Fig 4 **Scattering factor f_z for exponential axial distribution**

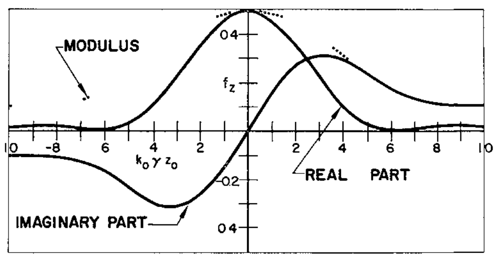

Fig 5 **Scattering factor f_z for linear axial distribution**

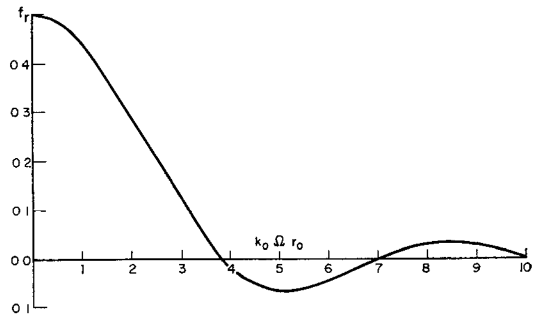

Fig 6 **Scattering factor f_r for homogeneous radial distribution**

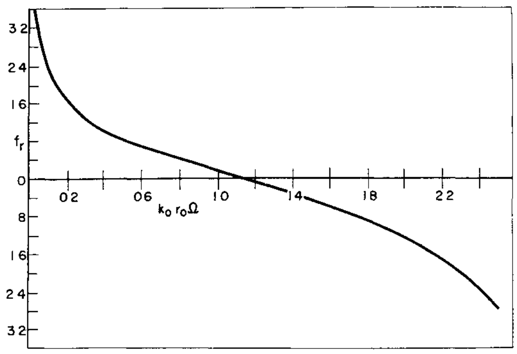

Fig 7 Scatteiing factor f_r for inveise-squaie ıadial distribution

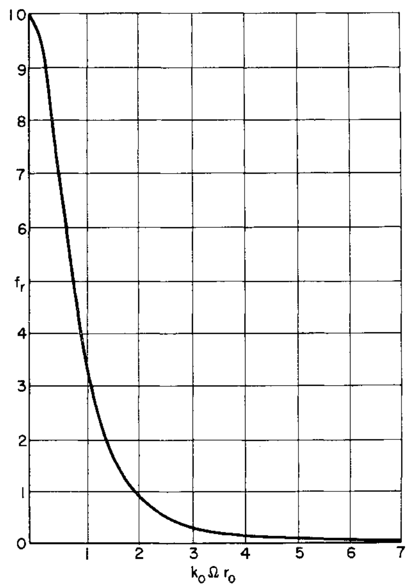

Fig 8 Scattering factoı f_r foı exponential ıadial distribution

siderably more severe than axial gradients, leading to the assumption of separable forms for η With these considerations in mind, some practical significance is attached to the results given below Note that in these cases the argument of a complex η will not change; only its modulus will vary

Figure 2 displays the factor Ω in the argument of the Bessel function (and hence all the radial functions) Specific functions can be inserted for A and B to determine their effects of the scattered fields To this end, the following forms are shown in the accompanying figures (Figs 3–10) from which 16 composite scattering regions can be constructed:

Axially Homogeneous (Fig 3)

$$A_1(V) = 1, 0 < V < 1; \; f(x) = i[1 - \exp(ix)]/x \quad (25)$$

Exponential Axial Decay (Fig 4)

$$A_2(V) = \exp(-V), V > 0; \; f(x) = (1 - ix)^{-1} \quad (26)$$

Linear Axial Decay (Fig 5)

$$A_3(V) = 1 - V, 0 < V \leq 1; \; f(x) = (i/x) + [1 - \exp(ix)]/x^2 \quad (27)$$

Radially Homogeneous (Fig 6)

$$B_1(U) = 1, 0 \leq U < 1; \; f(x) = J_1(x)/x \quad (28)$$

Radial Inverse-Square Decay (Fig 7)

$$B_2(U) = (1 + U^2)^{-1}, U \geq 0; \; f(x) = K_0(x) \quad (29)$$

Exponential Radial Decay (Fig 8)

$$B_3(U) = \exp(-U), U \geq 0; \; f(x) = (1 + x^2)^{-3/2} \quad (30)$$

Gaussian Radial Decay (Fig 9)

$$B_4(U) = \exp(-U^2), U \geq 0; \; f(x) = \tfrac{1}{2}\exp(-x^2/4) \quad (31)$$

Herlofson's Axial Model[18] (Fig 10)

$$A_4(V) = \exp(-V)[1 - \exp(-V)]^2, V \geq 0;$$
$$f(x) = \frac{x^2 + 3ix}{6(1 - x^2) + ix(x^2 - 11)} \quad (32)$$

Expressions (28) and (29) are Bessel functions in the notation of Ref 16

Discussion of the figures is deferred until the cross sections are displayed and the role of turbulence is introduced

Scattering and Absorption Cross Sections

The Poynting vector of the scattered field **S** is obtained directly from (17) Normalized with respect to the magni-

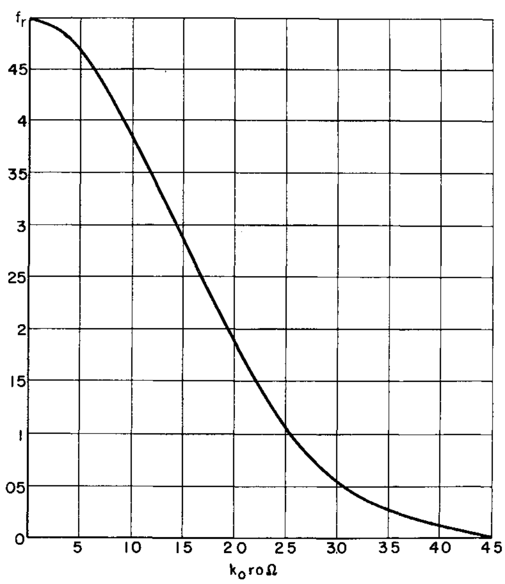

Fig 9 Scattering factor f_r for Gaussian ıadial distribution

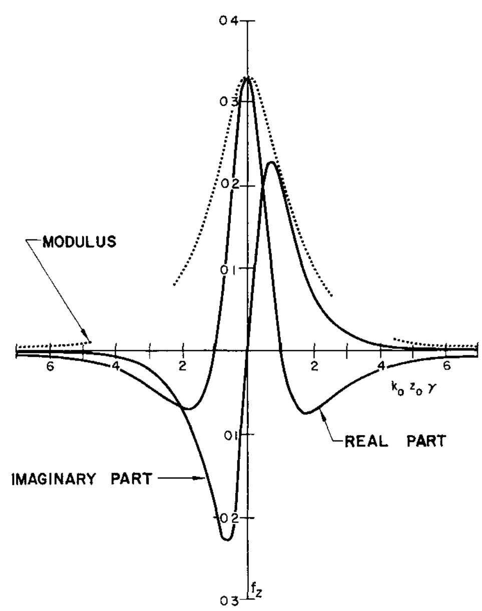

Fig 10 Scattering factor f_z for Herlofson's axial distribution

tude of the incident Poynting vector S_i the result is

$$\mathbf{S}_s = S_i \mathbf{e}_0 (k_0^2/4\pi r)^2 \sin^2\chi I(\eta) I^*(\eta) \tag{33}$$

where the asterisk implies complex conjugate, and χ is the angle between the incident electric field (polarization vector) and the line of sight to the point of observation Note that all dependence upon the direction of propagation of the illuminating wave (aspect angle) is confined to the arguments of the terms in the integral $I(\eta)$

The differential scattering cross section σ, defined as the ratio of the power scattered into a differential element of solid angle divided by the magnitude of the incident power flux, is given by

$$\sigma = r^2 S_s/S_i = (k_0^2 \sin\chi/4\pi)^2 I(\eta) I^*(\eta) \tag{34}$$

This simplifies in the separable case to the form

$$\sigma = (k_0^2 r_0^2 z_0 \sin\chi/2)^2 |\eta_0 f(k_0 \Omega r) f_z(k_0 \gamma z_0)|^2 \tag{35}$$

Note that $r_0^2 z_0$ can be considered a measure of the volume of the scattering zone The factor in vertical bars represents the influence of the geometrical distribution of the scattering material

The absorption cross section Q_a, defined as the power dissipated in the scattering zone per unit of incident power flux, can be computed in either of two ways Since the Born approximation permits the incident wave to propagate unimpeded through the scattering zone, the absorption cross section clearly must be given by

$$Q_a = \int\int\int [-k_0 Im(\eta)] \rho d\rho d\varphi dz = -2\pi k_0 \int\int Im(\eta) \rho d\rho dz \tag{36}$$

This result can be duplicated[4 7] by integrating the total radial Poynting vector over a sphere at infinity, neglecting terms of order η^2 which come from the scattered field The integral includes the incident Poynting vector, which vanishes on integration, the scattered Poynting vector, which is neglected, and the mixed term of field-field interaction The resulting integrals, being evaluated at infinity, achieve their values entirely from the points of stationary phase: $\theta_0 = \theta$, $\varphi_0 = 0$, and $\theta_0 = \pi - \theta$, $\varphi_0 = \pi$ At the first of these points, the factors Ω and γ in the function $I(\eta)$ vanish The second point contributes only imaginary quantities and hence does not indicate real power flow To illustrate this,

$$Q_a = -\frac{\omega\mu_0}{k_0} Re \left\{ \int_0^{2\pi} \int_0^{\pi} (\mathbf{E}_i \times \mathbf{H}^* + \mathbf{E} \times \mathbf{H}_i^*) \; \mathbf{e}_0 r^2 \sin\theta_0 d\theta_0 d\varphi_0 \right\} = -Re(I_1 + I_2) \tag{37}$$

where

$$I_1 = -\frac{k_0^2 r}{4\pi} \int_0^{2\pi} \int_0^{\pi} \sin^2\chi \exp[-ik_0 r(1 - \sin\theta \sin\theta_0 \cos\varphi_0 - \cos\theta \cos\theta_0)] I^*(\eta) \sin\theta_0 d\theta_0 d\varphi_0 \tag{38}$$

$$I_2 = -\frac{k_0^2 r}{4\pi} \int_0^{2\pi} \int_0^{\pi} (\mathbf{e}_i \; \mathbf{e}_0) \exp[ik_0 r(1 - \sin\theta \sin\theta_0 \cos\varphi_0 - \cos\theta \cos\theta_0)] I(\eta) \sin\theta_0 d\theta_0 d\varphi_0 \tag{39}$$

Note that the integrals are of the form

$$I = \int\int g(x,y) \exp[i\lambda f(x,y)] dx dy \tag{40}$$

where the range of integration includes a saddle point or col of $\exp(i\lambda f)$, say at (x_0,y_0) Assume now that $g(x,y)$ can be expanded about (x_0,y_0) as a slowly varying function, that $f(x,y)$ can similarly be expanded, and that λ is a very large parameter Noting that for the cases of interest here the mixed second partial of f vanishes at (x_0,y_0), a straightforward extension of the more familiar stationary phase arguments[19] leads to the following asymptotic form:

$$I \sim \frac{2\pi i}{\lambda} \frac{g(x_0,y_0) \exp[i\lambda f(x_0,y_0)]}{[(\partial^2 f/\partial x^2)_{(x_0 y_0)} (\partial^2 f/\partial y^2)_{(x_0 y_0)}]^{1/2}} \tag{41}$$

The point of stationary phase of interest here is $(\theta,0)$, as previously mentioned, at which point

$$\chi = \pi/2 \qquad \mathbf{e}_i = \mathbf{e}_0 \tag{42}$$

so that

$$I_1 = -\frac{k_0^2 r}{4\pi} \left\{ \frac{2\pi i}{-k_0 r} [\sin^2\theta(\sin^2\theta + \cos^2\theta)]^{-1/2} \right\} \sin\theta I^*(\eta)|_{\theta 0} = \frac{ik_0}{2} I^*(\eta)|_{\theta 0} \tag{43}$$

$$I_2 = -\frac{i}{2} k_0 I(\eta)|_{\theta 0} \tag{44}$$

Therefore

$$Q_a = -Re \left\{ \frac{ik_0}{2} [I^*(\eta) - I(\eta)]_{\theta 0} \right\} = -k_0 Im\{I(\eta)_{\theta 0}\} \tag{45}$$

and hence, from (18),

$$Q_a = -2\pi k_0 \int\int Im(\eta) \; \rho d\rho dz \tag{46}$$

in agreement with the expression (36) obtained from physical arguments ‡

The "radar cross section" σ_R is simply the differential cross section evaluated for backscatter,

$$\theta_0 = \pi - \theta \qquad \varphi_0 = \pi \tag{47}$$

at which point

$$\Omega = 2\sin\theta \qquad \gamma = 2\cos\theta \qquad \chi = \pi/2 \tag{48}$$

Thus, the radar cross section can be expressed, in general, as

$$\sigma_R = \frac{k_0^4}{4} \left| \int\int \eta(\rho,z) \exp(2ik_0 z \cos\theta) J_0(2k_0\rho \sin\theta) \; \rho d\rho dz \right|^2 \tag{49}$$

and for the separable case in the form

$$\sigma_R = (k_0^2 r_0^2 z_0/2)^2 \eta_0 \eta_0^* |f(2kr_0 \sin\theta) f(2k_0 z_0 \cos\theta)|^2 \tag{50}$$

Turbulent Wake Scattering

Since the major application for this class of scattering problems involves the high-speed flow of gases, a natural and necessary extension of the foregoing is generalization to include the random character of the electron distribution due to turbulence in the fluid stream This can be done in a straightforward manner, and an extensive literature exists in the general field[20] Much current work, however, concerns itself with the perennial debate over the structure of the turbulent spectrum[21] It is the intention here simply to illustrate the role of turbulence and deduce its fundamental effects on the observables

Restricting attention to those cases in which η is proportional to the local electron concentration, it is clear that a fluctuating value of electron density will give rise to a fluctuating scattered field Hence the best descriptor of the far-field intensities is an average or expected value Thus, from (15),

$$\langle \mathbf{E}_s \rangle = -k_0^2 [\mathbf{e}_p - \mathbf{e}_0(\mathbf{e}_0 \; \mathbf{e}_p)] \frac{\exp(ik_0 r)}{4\pi r} \times \int\int\int \langle \eta \rangle \exp[ik_0(\mathbf{e}_i - \mathbf{e}_0) \; \mathbf{r}'] d^3\mathbf{r}' \tag{51}$$

and all the succeeding expressions for field quantities hold if the expected value $\langle \eta \rangle$ is inserted for η This is not to say that the functions $A(z/z_0)$, $B(r/r_0)$ are to be considered

‡ It is not alarming that Q_a should be infinite for the "inverse square" radial distribution [Eq (29)] since there is an infinite volume of scattering material in this case

Table 1 Scattering factors t, t_z for various correlation function forms

$a(x)$	$t(y)$	$b(x)$	$t(y)$
$\exp(-\lvert x\rvert)$	$\dfrac{2}{1+y^2}$	$\exp(-x)$	$\dfrac{2\pi}{(1+y^2)^{3/2}}$
$\exp(-x^2)$	$\pi^{1/2}\exp(-x^2/4)$	$\exp(-x^2)$	$\pi\exp(-y^2/4)$
$1-\lvert x\rvert,\ \lvert x\rvert \leqslant 1$; $0,\ \lvert x\rvert > 1$	$\dfrac{4\sin y}{y} + \dfrac{\sin^2(y/2)}{(y/2)^2}$	$1-x^2,\ x \leqslant 1$; $0,\ x>1$	$\dfrac{4\pi}{y^2}J_2(y)$
$1-x^2,\ \lvert x\rvert \leqslant 1$; $0,\ \lvert x\rvert > 1$	$\dfrac{4}{y^3}(\sin y - y\cos y)$	$(1+x^2)^{-\nu}$ $(\nu>1)$	$\dfrac{2\pi}{\Gamma(\nu)}\left(\dfrac{y}{2}\right)^{\nu-1}K_{\nu-1}(y)$

probability distributions; they are, rather, the expected dis tributions The absorption cross section as well is also correct if $\langle\eta\rangle$ is used in the integral The scattering cross section, however, requires additional descriptors

Since σ can be written either as

$$\sigma = (k_0^2 \sin\chi/4\pi)^2 \left| \int\int\int \eta(\mathbf{r}') \exp[ik_0(\mathbf{e}_i - \mathbf{e}_0)\ \mathbf{r}']d^3\mathbf{r}' \right|^2 \tag{52}$$

or as an iterated integral,

$$\sigma = (k_0^2 \sin\chi/4\pi)^2 \int\int\int\int\int\int \eta(\mathbf{r})\eta^*(\mathbf{r}') \exp[ik_0(\mathbf{e}_i - \mathbf{e}_0)\ (\mathbf{r}-\mathbf{r}')]d^3\mathbf{r}d^3\mathbf{r}' \tag{53}$$

the expected value of σ involves a function of the fluctuations of η Specifically, writing η in the form

$$\eta = \langle\eta\rangle + \delta\eta \tag{54}$$

the expected value of σ is found to be

$$\langle\sigma\rangle = (k_0^2 \sin\chi/4\pi)^2\{\int\int [\langle\eta(\mathbf{r})\rangle\langle\eta^*(\mathbf{r}')\rangle + \langle\delta\eta(\mathbf{r})\delta\eta^*(\mathbf{r}')\rangle] \exp[ik_0(\mathbf{e}_i - \mathbf{e}_0)\ (\mathbf{r}-\mathbf{r}')]d^3\mathbf{r}d^3\mathbf{r}'\} \tag{55}$$

where the first term in the integral is simply what was previously computed, and the second represents a correction term due to turbulence Mathematically, it is the Fourier transform of the correlation function[15] of the fluctuations in η Physically, it must represent the product of the mean square of the fluctuations from average multiplied by a function that measures the region of coherence of these fluctuations If it is assumed that 1) the random field $\delta\eta$ is locally homogeneous, 2) the intensity of fluctuations in η is proportional to the local value of η, and 3) the correlation length is much smaller than the characteristic length for the mean distribution, then one may write, approximately,

$$\langle\delta\eta(\mathbf{r})\delta\eta^*(\mathbf{r}')\rangle \doteq \left|\eta\left(\frac{\mathbf{r}+\mathbf{r}'}{2}\right)\right|^2 I\Delta(\mathbf{r}-\mathbf{r}') \tag{56}$$

where I is the measure of the intensity of fluctuations of η and can be considered a normalizing factor for the correlation function Δ Choose

$$I = \langle\delta\eta(\mathbf{r})\delta\eta^*(\mathbf{r})\rangle/\eta(\mathbf{r})\eta^*(\mathbf{r}) \tag{57}$$

which is constant by assumption 2), and then

$$\Delta(0) = 1 \tag{58}$$

Changing variables, the second term of (55) can be written in factored form:

$$\langle\sigma\rangle_{\text{tu b}} = (k_0^2 \sin\chi/4\pi)^2 I(\int |\eta(R)|^2 d^3R) \times \{\int \Delta(\boldsymbol{\varrho}) \exp[ik_0(\mathbf{e}_i - \mathbf{e}_0)\ \boldsymbol{\varrho}]d^3\boldsymbol{\varrho}\} \tag{59}$$

Note that the last factor in (59) is of a form identical to the mean distribution integral in Eq (51) Thus the functional forms should be similar for the two integrals Also observe that only anisotropy in Δ can introduce angular dependence (other than the $\sin^2\chi$ factor) into the turbulent cross section To illustrate these points, assume that§

$$\Delta(\boldsymbol{\varrho}) = a(z/\zeta_0)b(r/\rho_0) \tag{60}$$

where ζ_0, ρ_0 are characteristic axial and radial dimensions of the correlation, respectively In this case, by analogy to Eq (24),

$$I(\Delta) = \int \Delta(\boldsymbol{\varrho}) \exp[ik_0(\mathbf{e}_i - \mathbf{e}_0)\ \boldsymbol{\varrho}]d^2\boldsymbol{\varrho} = \rho_0^2\zeta_0 t\,(k_0\Omega\rho_0)t\,(k_0\gamma\zeta_0) \tag{61}$$

where Ω and γ have the same meaning as before Table 1 displays functional forms of t_r and t_z for the corresponding distributions a and b

Discussion

The relative importance of the turbulent contribution can now be assessed For example, consider the backscatter cross section σ_R when the scattering region can be described under the forementioned assumptions, and the mean dis tribution of η is separable In this case,

$$\langle\sigma_R\rangle = (k_0^2r_0^2z_0/2)^2\eta_0\eta_0^*|f\,(2k_0r_0\sin\theta)f\,(2k_0z_0\cos\theta)|^2 + \frac{I}{8\pi}k_0^4r_0^2z_0\rho_0^2\zeta_0\eta_0\eta_0^*[\int A(x)dx] \times [\int B(y)ydy]t\,(2k_0\rho_0\sin\theta)t\,(2k_0\zeta_0\cos\theta) \tag{62}$$

The order of magnitude of the ratio of the turbulent con tribution to the "mean" contribution is thus given by

$$\frac{\rho_0^2\zeta_0}{r_0^2z_0}\ \frac{t_r(2k_0\rho_0\sin\theta)t_z(2k_0\zeta_0\cos\theta)}{|f\,(2k_0r_0\sin\theta)f\,(2k_0z_0\cos\theta)|^2} \tag{63}$$

which, regardless of the values of ρ_0 and ζ_0 (other than zero), grows larger as r_0 and/or z_0 increase beyond critical values of the order of $(2k_0\sin\theta)^{-1}$ or $(2k_0\cos\theta)^{-1}$ This is because of the rapid decay of the denominator factors f^2 and f_z^2, corresponding physically to the decrease of reflective gradients in the dielectric constant as the scale of the mean distribution increases

In other words, if the scattering zone is large compared to a wavelength, it encompasses a large number of incoherently scattering subzones (on the average) which provide a greater return than that due to the "mean" distribution which con tributes only weak gradients in $\langle\eta\rangle$ Thus the transition to turbulent flow in the wake of a projectile may be marked by a sharp increase in radar cross section [22] This observation clearly is not dependent upon the specific functions and separability approximations invoked here, but is a generalization one can abstract from the specific illustrations presented here

Note that, under the separability assumption for Δ [Eq (60)], the figures (Figs 3–10) presented for scattering by the

§ This separation is thought not to be unreasonable, as it reflects a basic feature of the wake flow, i e , diffusive transport in the radial direction and convective as well as diffusive transport in the axial direction The specific functional forms chosen are for convenience of illustration and do not rest upon consideration of the physical processes mean distribution are applicable if the functions are evaluated for $\gamma z_0 = |\gamma\zeta_0|$, $r_0 = \rho_0$, and the value of f is doubled A factor of 2π should then be inserted on the right-hand side of (61) for consistency

It should be pointed out that, in evaluating experimental data in which the aspect angle changes significantly, even rather weak anistropy can contribute to relatively large changes in σ_R This is evidenced if one assumes that the functional forms for a and b are the same and have the same characteristic length, of the order of a wavelength The function t is then quite sensitive near $\theta = \pi/2$

The dependence of σ on wavelength might provide clues to the functional forms for f_r and f (or t_r and t), since the coefficients of these factors are independent of wavelength in this approximation, as η varies like k_0^{-2}

Appendix

It is desired to carry out the angular portion of

$$I(\eta) = \int\int\int \eta \exp[ik_0(\mathbf{e}_i - \mathbf{e}_0) \mathbf{r}]d^3\mathbf{r}'$$

$$= \int\int\int_0^{2\pi} \eta(\rho',z') \exp[ik_0(a\rho' \sin\varphi' + b\rho' \cos\varphi' + \gamma z')]d\varphi'\rho'd\rho'dz' \quad \text{(A1)}$$

where

$$a = -\sin\theta_0 \sin\varphi_0 \quad \text{(A2)}$$

$$b = \sin\theta - \sin\theta_0 \cos\varphi_0 \quad \text{(A3)}$$

$$\gamma = \cos\theta - \cos\theta_0 \quad \text{(A4)}$$

Hence[23]

$$I(\eta) = 2\pi \int\int \eta(\rho',z') \exp(ik_0\gamma z')J_0(k_0\rho'\Omega)\rho'd\rho'dz' \quad \text{(A5)}$$

$$(a^2 + b^2)^{1/2} = \Omega = (\sin^2\theta_0 + \sin^2\theta - 2 \sin\theta \sin\theta_0 \cos\varphi_0)^{1/2} \quad \text{(A6)}$$

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